§ 5.5 Complex Eigenvalues

We have seen before that the eigenvalues of an $n \times n$ matrix are the roots of its characteristic polynomial. Since this a polynomial of degree n, if we extend it into the complex numbers $\mathbb{C}$, it will have exactly n roots (counting multiplicity) and hence n eigenvalues (counting multiplicity).

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and diagonalize it if possible.

Solution

The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$$

which is zero at $\lambda = \pm i$, thus the eigenvalues are

$$\boxed{\lambda_1 = i \quad \text{and} \quad \lambda_2 = -i}$$

(Notice A is 2×2 and has 2 distinct eigenvalues so it must be diagonalizable!)

To find the eigenvectors we look for a basis of the eigenspaces.

- $A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$

  $\Rightarrow$ Null $(A - iI)$ has basis $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$

- $A - (-i)I = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$

  $\Rightarrow$ Null $(A - (-i)I)$ has basis $\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$

Thus

- $v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is an eigenvector associated to $\lambda_1 = i$

- $v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ is an eigenvector associated to $\lambda_2 = -i$

Moreover $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ can be diagonalized as

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$$

Remark

- Factoring polynomials over $\mathbb{C}$ is difficult in general.

- Recall the quadratic formula: The solutions to $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Theorem (#7 handwritten HW 26)

If $p(x)$ is a polynomial with real coefficients and z is a complex number with $p(z) = 0$, then $p(\bar{z}) = 0$ as well.

As a consequence, if A is an $n \times n$ matrix with real entries and λ is an eigenvalue, then $\bar{\lambda}$ is as well.

(verify this on the previous example)

This is nice, but what does it mean for the corresponding eigenvectors?

Defn

Let x be a vector in $\mathbb{C}^n$. The <u>real part</u> and <u>imaginary part</u> of x are the vectors of $\mathbb{R}^n$ obtained from the real and imaginary parts of the entries of x, i.e.

$$x = \operatorname{Re} x + i \operatorname{Im} x$$

Example

Let $x = \begin{bmatrix} 3-i \\ i \\ 2+5i \end{bmatrix}$ in $\mathbb{C}^3$, then

$$x = \underbrace{\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}}_{\operatorname{Re} x} + i \underbrace{\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}}_{\operatorname{Im} x}$$

We define the <u>conjugate</u> of vector x as the vector obtained from the complex conjugates of each entry of x.

Example

If $x = \begin{bmatrix} 3-i \\ i \\ 2+5i \end{bmatrix}$ then $\overline{x} = \begin{bmatrix} 3+i \\ -i \\ 2-5i \end{bmatrix}$

Theorem

Let A be an $n \times n$ matrix with real entries. If λ is an eigenvalue of A and v is an eigenvector corresponding to λ, then $\overline{\lambda}$ is an eigenvalue and $\overline{v}$ is an eigenvector corresponding to it.

(verify this on first example)

Real matrices acting on $\mathbb{C}^2$

Let $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ be a 2×2 matrix where a and b are real numbers. The eigenvalues of C are $\lambda = a \pm bi$ (exercise: show this)

Now looking at $\lambda = a + bi$, let

$$r = |\lambda| = \sqrt{a^2+b^2} \quad \text{and} \quad \theta = \arg \lambda = \tan^{-1}\left(\frac{b}{a}\right)$$

then we can write

$$C = r \cdot \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scaling by a factor of } r} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation by } \theta}$$

Thus any such matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ acts on $\mathbb{C}^2$ as a composition of rotation by θ and then scaling by a factor of r.

Theorem

Let A be a real 2x2 matrix with complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and associated eigenvector v in $\mathbb{C}^2$. Then $A = PCP^{-1}$ where

$$P = \left[\operatorname{Re} v \,\middle|\, \operatorname{Im} v \right] \quad \text{and} \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$